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THE BEHAVIOUR OF AN INVERTIBLE MECHANICAL SYSTEM ON THE BOUNDARY OF THE STABILITY REGION†

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A critical case is shown to occur on the boundary of the stability region of an invertible mechanical system: the characteristic equation has two zero roots corresponding to a single group of solutions, the other roots being purely imaginary. The system is then as a rule unstable.

1. STATEMENT OF THE PROBLEM

WE WILL be concerned with the stability of the equilibrium position of a mechanical system under the action of position forces and forces which are quadratic functions of the velocities

$$\frac{d}{dt}\frac{\partial T}{\partial q_s^{\cdot}}-\frac{\partial T}{\partial q_s}=Q_s(\mathbf{q})+\sum_{i,j}f_{sij}(\mathbf{q})\,q_i^{\cdot}q_j^{\cdot}$$
$$2T=\sum_{i,j}a_{ij}(\mathbf{q})\,q_i^{\cdot}q_j^{\cdot} \tag{1.1}$$

where T is the kinetic energy and Q_s, a_{ij}, f_{sij} are holomorphic functions of $\mathbf{q}$. Throughout this paper, unless otherwise stated, $j, s=1, 2, \ldots, n$ and summation over the indices i, j, k, l is performed from 1 to n. On the assumption that an equilibrium position corresponds to zero values of the coordinates, $q_s^0=0$ and $Q_s(\mathbf{0})=0$, let us solve system (1.1) for the highest-order derivatives

$$q_s^{\cdot\cdot}=\sum_j b_{sj}q_j+F_s(\mathbf{q})+\sum_{j,k}c_{sjk}(q)\,q_j^{\cdot}q_k^{\cdot} \tag{1.2}$$

Here F_s and c_{sjk} are holomorphic functions of $q_1, \ldots, q_n$ and the expansions of the functions $F_s(\mathbf{q})$ begin with second-order terms in $\mathbf{q}$; b_{sj} are constants.

The characteristic equation

$$\Delta(\varkappa^2)=\det\|b_{sj}-\delta_{sj}\varkappa^2\|=0 \tag{1.3}$$

has only even powers of $\varkappa$. Consequently, if at least one of the roots of (1.3), $\varkappa^2=\lambda_1^2, \ldots, \lambda_n^2$, is positive, then the equilibrium position is unstable to a first approximation [1]. In the parameter space of the system, therefore, the conditions $\lambda_s^2<0$ define the stability region (to a first approximation). The present author studied this system on the assumption that all the roots λ_s^2 are negative. On the boundary of the stability region at least one of the numbers λ_s^2 vanishes.

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Let us assume that $\lambda_1^2 = 0$, $\lambda_i^2 < 0$ $(j = 2, \ldots, n)$ and bring the linear approximation of the system to canonical form. In new coordinates

$$x = 2i \sum_j p_{1j} q_j, \quad y = 2 \sum_j p_{1j} q_j^{\cdot}, \quad z_s = \sum_j p_{sj} (q_j^{\cdot} + \lambda_s q_j), \tag{1.4}$$
$$\bar{z}_s = \sum_j p_{sj} (-q_j^{\cdot} + \lambda_s q_j) \quad (s = 2, \ldots, n)$$

($\bar{z}_s$ is the complex conjugate of z_s) the linear system becomes

$$x^{\cdot} = iy, \; y^{\cdot} = 0, \; z_s^{\cdot} = \lambda_s z_s, \; \bar{z}_s^{\cdot} = -\lambda_s \bar{z}_s \; (s = 2, \ldots, n)$$

if the purely imaginary constants p_{sj} are determined from the following systems of linear equations

$$\begin{aligned} (b_{11} - \lambda_s^2) p_{s1} + b_{21} p_{s2} + \ldots + b_{n1} p_{sn} &= 0 \\ b_{12} p_{s1} + (b_{22} - \lambda_s^2) p_{s2} + \ldots + b_{n2} p_{sn} &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ b_{1n} p_{s1} + b_{2n} p_{s2} + \ldots + (b_{nn} - \lambda_s^2) p_{sn} &= 0 \end{aligned} \tag{1.5}$$

Since λ_s^2 are the roots of the characteristic equation (1.3), the determinants of systems (1.5) vanish and so these systems have non-trivial solutions. Obviously, if no two of the numbers λ_j^2 $(j = 2, \ldots, n)$, are equal, we have $\det\|p_{sj}\| \neq 0$, which immediately implies that the transformation (1.4) is non-singular. The determinant of its matrix is equal to

$$i 2^{n-1} \prod_{j=2}^{n} \lambda_j \{\det \| p_{sj} \| \}^2$$

Let us express q_s, $q_s^{\cdot}$ in terms of the new variables (1.4). We have

$$z_s + \bar{z}_s = 2\lambda_s \sum_j p_{sj} q_j, \quad z_s - \bar{z}_s = 2 \sum_j p_{sj} q_j^{\cdot} \quad (s = 2, \ldots, n)$$

whence (d_{sj} are purely imaginary constants)

$$q_s = \frac{1}{2} \sum_j \frac{d_{sj}}{\varkappa_j} \xi_j, \quad q_s^{\cdot} = \frac{1}{2} \sum_j d_{sj} \eta_j \tag{1.6}$$

$$\varkappa_1 = i, \; \varkappa_k = \lambda_k, \; \xi_1 = x, \; \xi_k = z_k + \bar{z}_k, \; \eta_1 = y, \; \eta_k = z_k - \bar{z}_k \; (k = 2, \ldots, n)$$

We will now write the result of the transformation to the new variables

$$\begin{gathered} x^{\cdot} = iy, \; y^{\cdot} = Y(x, y, \mathbf{z}, \bar{\mathbf{z}}) \\ \mathbf{z}^{\cdot} = \Lambda \mathbf{z} + \mathbf{Z}(x, y, \mathbf{z}, \bar{\mathbf{z}}), \; \bar{\mathbf{z}}^{\cdot} = -\Lambda \bar{\mathbf{z}} + \overline{\mathbf{Z}}(x, y, \mathbf{z}, \bar{\mathbf{z}}) \end{gathered}$$

$$\begin{gathered} \mathbf{z} = (z_2, \ldots, z_n), \; \mathbf{Z} = (Z_2, \ldots, Z_n), \; \Lambda = \operatorname{diag}(\lambda_2, \ldots, \lambda_n) \\ Y = 2 \sum_j p_{1j} \Big[F_j(\mathbf{q}) + \sum_{k,l} c_{jkl}(\mathbf{q}) q_k^{\cdot} q_l^{\cdot}\Big]_{(1.6)} \\ Z_s = \sum_j p_{sj} \Big[F_j(\mathbf{q}) + \sum_{k,l} c_{jkl}(\mathbf{q}) q_k^{\cdot} q_l^{\cdot}\Big]_{(1.6)} \end{gathered} \tag{1.7}$$

where the bar denotes complex conjugation and the linear approximation is written down explicitly.

By (1.7), the expansions of the functions, Y, $\mathbf{Z}$, $\bar{\mathbf{Z}}$ in terms of powers of x, y, $\mathbf{z}$, $\bar{\mathbf{z}}$ have only purely imaginary coefficients. This is not surprising, because the original system (1.1) is invertible with a linear automorphism M: $t \to -t$, $\mathbf{q} \to \mathbf{q}$, $\mathbf{q}^\cdot \to -\mathbf{q}^\cdot$, which under the linear transformation (1.4) becomes

$$t \to -t, \quad x \to x, \quad y \to -y, \quad \mathbf{z} \to \bar{\mathbf{z}}, \quad \bar{\mathbf{z}} \to \mathbf{z}$$

2. STABILITY

It follows from our exposition up to now that on the boundary of the stability region one obtains the critical case of two zero roots with a single group of solutions and $n-1$ pairs of purely imaginary roots. The stability problem in this case was solved in the case when $n = 1$ by Lyapunov [1] and later by Kamenkov [4]. Henceforth, for convenience, we replace $n-1$ by n and consider a real system

$$\begin{gathered} x^\cdot = y, \quad y^\cdot = Y(x, y, \mathbf{u}, \mathbf{v}) \\ u_s^\cdot = \omega_s v_s + U_s(x, y, \mathbf{u}, \mathbf{v}), \quad v_s^\cdot = -\omega_s u_s + V_s(x, y, \mathbf{u}, \mathbf{v}) \end{gathered} \tag{2.1}$$

In so doing we must replace y in system (1.7) by iy, $\lambda_s = i\omega_s$, and u_s and v_s are the real and imaginary parts, respectively, of z_s. Instead of the automorphism M we have N

$$t \to -t, \quad x \to x, \quad y \to -y, \quad \mathbf{u} \to \mathbf{u}, \quad \mathbf{v} \to -\mathbf{v}$$

Applying this automorphism and the transformation formulae (1.4), we deduce from system (2.1) that

$$U_s(x, 0, u, \mathbf{0}) \equiv 0, \quad \partial V_s/\partial y|_* = \partial V_s/\partial v_j|_* = \partial Y/\partial y|_* = \partial Y/\partial v_j|_* \equiv 0 \tag{2.2}$$

We now subject system (2.1) to a change of variables

$$u_s = u_s^* + f_s(x), \quad v_s = v_s^* + y\theta_s(x) \tag{2.3}$$

where the unknown functions f_s, θ_s are determined from the simultaneous systems of functional equations

$$-\omega_s f_s(x) + V_s(x, 0, \mathbf{f}(x), \mathbf{0}) = 0 \tag{2.4}$$

$$\omega_s \theta_s(x) + \partial U_s/\partial y\,|_{y=0,\, \mathbf{u}=\mathbf{f}(x),\, \mathbf{v}=0} = f_s'(x) \tag{2.5}$$

(the prime denotes differentiation with respect to x). Then the following conditions must also hold together with (2.2)

$$V_s(x, 0, \mathbf{0}, \mathbf{0}) = -\theta_s Y(x, 0, \mathbf{0}, \mathbf{0}), \quad \theta_s(0) = 0, \quad \partial U_s/\partial y\,|_* = 0 \tag{2.6}$$

A second change of variable

$$y = y^* + \sum_j v_j^* \varphi_j(x) \tag{2.7}$$

where the unknown functions $\varphi j(x)$ are determined from the system of linear equations

$$-\omega_s \varphi_s(x) + \sum_j \partial V_j^*/\partial u_s^*\,|_*\, \varphi_j(x) = \partial Y^*/\partial u_s^*\,|_*$$

[Y^* denotes the function Y after the substitution (2.3)], will reduce the equation for y to a form in which

$$\partial Y/\partial u_j\,|_* = 0 \tag{2.8}$$

Conditions (2.2) and (2.6) remain valid.

Finally, a last transformation

$$\xi_s = u_s^* + \sum_j f_{sj}(x)\,u_j^*, \qquad \eta_s = v_s^* + \sum_j \theta_{sj}(x)\,v_j^* \tag{2.9}$$

with as yet undetermined functions $f_{sj}(x)$, $\theta_{sj}(x)$, leads to the system

$$\xi_s^{\cdot} = \omega_s\eta_s - \omega_s\sum_j \theta_{sj}v_j^* + \sum_j f_{sj}(\omega_j v_j^* + U_j^*) + U_s^* + \sum_j f_{sj}'(x)\,u_j^* y$$
$$\eta_s^{\cdot} = -\omega_s\xi_s + \omega_s\sum_j f_{sj}u_j^* + \sum_j \theta_{sj}(-\omega_j u_j^* + V_j^*) + V_s^* + \sum_j \theta_{sj}'(x)\,v_j^* y$$

For each fixed s we determine the functions f_{sj}, θ_{sj} $(j\neq s)$ from the system

$$-\omega_s\theta_{sj} + \omega_j f_{sj} + \sum_{k\neq s} \partial U_k^*/\partial v_j^*\,|_* f_{sk} + \partial U_s^*/\partial v_j^*\,|_* = 0$$
$$-\omega_j\theta_{sj} + \omega_s f_{sj} + \sum_{k\neq s} \partial V_k^*/\partial u_j^*\,|_* \theta_{sk} + \partial V_s^*/\partial u_j^*\,|_* = 0 \quad (s, j = 1,\ldots, n;\ j\neq s) \tag{2.10}$$

By conditions (2.2), after the substitutions (2.9) and (2.10), system (2.1), with conditions (2.2), (2.6) and (2.8), will also satisfy the condition

$$\partial U_s/\partial v_j\,|_* \equiv 0, \quad \partial V_s/\partial u_j\,|_* \equiv 0 \quad (s\neq j) \tag{2.11}$$

The reader will note that all the transformations preserve the automorphism N.

As a result of transformations (2.3), (2.7) and (2.9), we obtain

$$x^{\cdot} = y + \sum_j v_j\varphi_j(x),$$
$$y^{\cdot} = Y_0(x) + \sum_{j,k=1}^{2n+1} [\mathring{Y}_{jk}(x) + Y_{jk}(x, y, \mathbf{u}, \mathbf{v})]\,w_j w_k$$
$$u_s^{\cdot} = [\omega_s + \mu_s(x)]\,v_s + \sum_{j,k=1}^{2n+1} [\mathring{U}_{sjk}(x) + U_{sjk}(x, y, \mathbf{u}, \mathbf{v})]\,w_j, w_k$$
$$v_s^{\cdot} = -[\omega_s + \nu_s(x)]\,u_s + V_{s0}(x) + \sum_{j,k=1}^{2n+1} [\mathring{V}_{sjk}(x) + V_{sjk}(x, y, \mathbf{u}, \mathbf{v})]\,w_j w_k \tag{2.12}$$

Here w denotes y, $\mathbf{u}$, $\mathbf{v}$ and the functions Y_{jk}, U_{sjk}, V_{sjk} vanish when $x = y = 0$, $\mathbf{u} = \mathbf{v} = 0$. In addition

$$V_{s0}(x) = -\theta_s(x)\,Y_0(x), \quad \theta_s(0) = 0 \tag{2.13}$$

Consider the function

$$V = xW, \quad W = (1 + \alpha x)\,y^2 - \beta\sum_s \{[1 + \nu_s(x)]\,u_s^2 + [1 + \mu_s(x)]\,v_s^2\}$$

where α and β are certain constants. The derivative of this function along trajectories of system (2.12) is

$$V^{\cdot} = \left[y + \sum_j v_j \varphi_j(x)\right] W + xW^{\cdot}$$

$$W^{\cdot} = \Phi + 2\alpha xyY_0(x) - 2\beta \sum_s [1 + \mu_s(x)] v_s V_{s0}(x) +$$

$$+ \sum_{s,j,k=1}^{2n+1} H_{sjk}(x, y, \mathbf{u}, \mathbf{v}) w_s w_j w_k, \qquad H_{sjk}(0, 0, 0, 0) = 0$$

$$\Phi = \alpha y^2 \left[y + \sum_j v_j \varphi_j(x)\right] + 2y\left[Y_0(x) + \sum_{j,k=1}^{2n+1} \mathring{Y}_{jk}(0) w_j w_k\right] -$$

$$- 2\beta \sum_s \sum_{j,k=1}^{2n+1} \{[1 + \nu_s(0)] \mathring{U}_{sjk}(0) u_s + [1 + \mu_s(0)] \mathring{V}_{sjk}(0) v_s\} w_j w_k$$

Suppose that in the neighbourhood in question $|x| \leq \delta$ and

$$\max_{|x| \leq \delta} \{|\varphi_j(x)|, |\mu_j(x)|, |\nu_j(x)|\} = \varepsilon, \quad \max \{|Y_{jk}{}^0(0)|, |U^0_{sjk}(0)|, |V^0_{sjk}(0)|\} = A$$

Since the functions $\boldsymbol{\varphi}(x)$, $\boldsymbol{\mu}(x)$, $\boldsymbol{\nu}(x)$ vanish at zero, it follows that for small δ the number ε is also small. Choose positive α, β so that in the domain $W>0$, $y>0$

$$y + \sum_j v_j \varphi_j(x) > y/2, \qquad W^{\cdot} > \Psi = y^3 + yY_0(x) \tag{2.14}$$

In this domain

$$|u_s| < \gamma y, \quad |v_s| < \gamma y \quad (s = 1, \ldots, n), \quad 0 < \gamma = \sqrt{\frac{1 + \alpha\delta}{\beta(1 - \varepsilon)}} < 1 \tag{2.15}$$

and conditions (2.14) will hold if

$$2n\varepsilon\gamma < 1, \quad \alpha > 4(2n + 1)^2 (1 + 2\beta) A + 4$$

In fact, by (2.12) and (2.15) and the fact that the functions Y_{jk}, U_{sjk}, V_{sjk} vanish at $x = y = 0$, $\mathbf{u} = \mathbf{v} = 0$ in the domain $W>0$, $y>0$, all the terms in the expression for $W^{\cdot}$, except Φ, are $o(y^3, yY_0(x))$, while $\Phi > 2\Psi$ for sufficiently small x. Therefore, if $V>0$ in the domain $x>0$, $y>0$, $W>0$ where $\Psi>0$, it follows from Chetayev's Instability Theorem [5] that the system is unstable.

Let

$$Y_0(x) = gx^m + \ldots \; (g = \text{const}) \tag{2.16}$$

Obviously, if m is even, we can always ensure by substituting x for $-x$ and y for $-y$ that the coefficient g is positive. If $g>0$ then $\Psi>0$ in the domain $V>0$. Thus, the system is unstable for even m, while if m is odd it is unstable if $g>0$. But if $Y_0(x) \equiv 0$, the sign of Ψ is determined by $y^3>0$ and we again obtain instability.

Theorem. The trivial solution of system (2.12), (2.13) and (2.16) is unstable in Lyapunov's sense in each of the following cases: (a) m is even; (b) m is odd, $g>0$; (c) $Y_0(x) \equiv 0$.

We see, then, that system (1.1) is generally unstable on the boundary of its stability region. It follows from the form of Chetayev's function we have constructed that an increasing solution necessarily implies an increase in x. Therefore, by the transformation formulae (1.4), one of the coordinates q must also increase along such a solution.

The conclusions of the theorem are independent of the number n of purely imaginary roots. The decisive elements in this situation are the zero roots, which are indeed responsible for the instability.

3. EXAMPLE. MODEL OF AN ELASTIC ROD WITH AN APPLIED TRACKING FORCE [6]

Consider a mechanical system comprising two identical rods of mass m and length l, attached by a hinge and spiral spring of stiffness c_2 and placed on a smooth horizontal plane. The end of the first rod is attached by a hinge and a helical spring of stiffness c_1 to a fixed point; the free end of the other rod is subjected to a tracking force F directed along the axis of the rod. In the underformed state of the spring both rods lie along a straight line—the x axis.

We take as generalized coordinates the angles φ_1 and φ_2 by which the rods deviate from the x axis. Then in system (1.1)

$$T = {}^1\!/_6 ml^2 [4\varphi_1^{\cdot 2} + 3\varphi_1^{\cdot}\varphi_2^{\cdot} \cos(\varphi_2 - \varphi_1) + \varphi_2^{\cdot 2}], \quad f_{sij}(\mathbf{q}) \equiv 0$$

$$Q_1 = -c_1\varphi_1 + c_2(\varphi_2 - \varphi_1) - Fl \sin(\varphi_2 - \varphi_1), \quad Q_2 = c_2(\varphi_1 - \varphi_2)$$

we have a mechanical system under the action of potential and non-conservative position forces. Noting equations (1.2), we write the characteristic equation (1.3) as

$$\varkappa^4 + \varkappa^2\omega^2 + \frac{36}{7}\frac{c_1c_2}{ml^2} = 0, \quad \omega^2 = \frac{6}{7ml^2}(2c_1 + 16c_2 - 5Fl)$$

which has two pairs of purely imaginary roots provided that

$$2c_1 + 16c_2 - 5Fl - 2\sqrt{7c_1c_2} > 0 \quad (c_1c_2 \neq 0)$$

This case was studied in [2]. But if $c_1c_2 = 0$, we get

$$\lambda_1^2 = 0, \ \lambda_2^2 = -\omega^2, \ \omega^2 > 0$$

and our transformation (1.4) is here

$$x = -2\frac{\omega^2}{b_{12}}\left[\frac{b_{21}}{b_{11}}\varphi_1 - \varphi_2\right], \quad y = 2i\frac{\omega^2}{b_{12}}\left[\frac{b_{21}}{b_{11}}\varphi_1^{\cdot} - \varphi_2^{\cdot}\right]$$

$$z = -i\frac{\omega^2}{b_{12}}\left[-\frac{b_{22}+\omega^2}{b_{12}}(\varphi_1^{\cdot} + i\omega\varphi_1) + (\varphi_2^{\cdot} + i\omega\varphi_2)\right]$$

$$b_{11} = -2b_0 - b_{12}, \ b_{21} = 3b_0 - b_{22}$$

$$b_{12} = \frac{30c_2 - 12Fl}{7ml^2}, \quad b_{22} = \frac{-66c_2 + 18Fl}{7ml^2}, \quad b_0 = \frac{6c_1}{7ml^2}$$

The inverse transformation (1.6) is

$$\varphi_1 = \frac{1}{2}\left(x - \frac{z+\bar z}{\omega}\right), \quad \varphi_2 = \frac{1}{2}\left(\frac{b_{22}+\omega^2}{b_{12}}x - \frac{b_{21}}{b_{11}}\frac{z+\bar z}{\omega}\right)$$

$$\varphi_1^{\cdot} = \frac{1}{2i}(-y + z - \bar z), \quad \varphi_2^{\cdot} = \frac{1}{2i}\left[-\frac{b_{22}+\omega^2}{b_{12}}y + \frac{b_{21}}{b_{11}}(z - \bar z)\right]$$

Expand the right-hand sides of the equations of perturbed motion in power series in terms of φ and $\varphi^{\cdot}$. It turns out that there are not second-order terms on the right of (1.2), while the fourth-order terms are

$$\Phi_1^{(3)} = \xi(\varphi_2 - \varphi_1)^3 + \frac{1}{7}(9\varphi_1^{\cdot 2} + 6\varphi_2^{\cdot 2})(\varphi_2 - \varphi_1)$$

$$\Phi_2^{(3)} = -\xi(\varphi_2 - \varphi_1)^3 - \frac{1}{7}(24\varphi_1^{\cdot 2} + 9\varphi_2^{\cdot 2})(\varphi_2 - \varphi_1), \quad \xi = \frac{12Fl - 9c_2}{7ml^2}$$

Thus the expansion of the function $Y_0(x)$ in system (2.12) begins with fourth-order terms. It is obvious from the transformation formulae that

$$g = -\omega^2\xi\,\frac{b_{11}+b_{21}}{4b_{11}}\left[\frac{b_{22}+\omega^2}{b_{12}}-1\right]^3$$

Hence we have

$$b_{11}+b_{21} = \frac{6c_1+36c_2-6Fl}{7ml^2}, \quad b_{22}+\omega^2-b_{12} = \frac{12c_1-96c_2}{7ml^2}$$

Let $c_2 = 0$. Then if $2c_1 - 5Fl > 0$ $(\omega^2 > 0)$ we have $b_{11}+b_{21}>0$, $b_{22}+\omega^2-b_{12}>0$, $b_{11}<0$, $b_{12}>0$ and the equilibrium position is unstable.

Let $c_1 = 0$. Then since $\omega^2>0$, we have $16c_2-5Fl>0$. Here $b_{11}+b_{21}>0$, $b_{22}+\omega^2-b_{12}<0$, $b_{11}b_{12}<0$ and if $Fl<3/4c_2$ we have $g>0$; the equilibrium is unstable.

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